Positively-homogeneous Konüs-Divisia indices and their applications to demand analysis and forecasting

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Abstract. This paper is devoted to revealed preference theory and its applications to testing economic data for consistency with utility maximization hypothesis, construction of index numbers, and forecasting. The quantitative measures of inconsistency of economic data with utility maximization behavior are also discussed. The structure of the paper is based on comparison between the two tests of revealed preference theory – generalized axiom of revealed preference (GARP) and homothetic axiom of revealed prefernce (HARP). We do this comparison both theoretically and empirically. In particular we assess empirically the power of these tests for consistency with maximization behavior and the size of forecasting sets based on them. For the forecasting problem we show that when using HARP there is an effective way of building the forecasting set since this set is given by the solution of the system of linear inequalities. The paper also touches upon the question of testing a set of Engel curves rather than finite set of observations for consistency with utility maximization behavior and shows that this question has effective solution when we require the rationalizing utility function to be positively homogeneous.

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1 Introduction

Statistical services in various countries compute economic indices using the approach based on the value of consumption bundle in different time periods. The problem of this approach is that the consumption bundle changes with the change of the structure of prices. The statistical services compute Laspeyres and Paasche price indices[1] fixing some consumption bundle equal to consumption levels in base or current period. More

[1]See (Diewert, 2004), for example

precisely, for Laspeyres index they take the set of goods consumers bought in base period as consumption bundle. For Paasche index consumption in current period is taken as the consumption bundle. Usually the value of Paasche index is not greater than that of Laspeyres index. This systematic difference is called Gerschenkron's effect. It is caused by substitution effects – when a price on certain commodity increases people tend to substitute it with different goods. This implies that consumption structure is changing and no fixed consumption bundle can represent what actually consumers buy. See (Ershov, 2011; International Labor Office, 2004) for a detailed review of existing approaches to the problem of index numbers construction.

We address the issue of substitution effects by suggesting a new approach to index numbers. The approach is based on Pareto's theory of demand. According to this theory a representative economic agent is supposed to choose his or her consumption bundle among inter-substitutable goods by maximizing his or her utility function given his or her budget constraint. Let us call the problem of construction demand functions from utility function as a direct problem. Then the inverse one is to recover utility function from demand functions. Solvability of inverse problem is related to integrability of demand functions. This problem is central for revealed preference theory.

An approach for construction of consumption index based on the solution of inverse problem was suggested in (Konüs, 1924)[2]. This index is called Konüs consumption index. An effective way of computing Konüs consumption index is based on Afriat's contributions to revealed preference theory (see (Afriat, 1967)).

Divisia index numbers formulas, first introduced in (Divisia, 1925), represent a general form of consumption and price indices. See (Balk, 2005) for further details, systematic description and a historical overview of Divisia index numbers. In general, Konüs consumption index is not a Divisia index. However, when we add the requirement for the utility function to be positive-homogeneous of degree 1, Konüs consumption index becomes Divisia index (Hulten, 1973) and is called Konüs-Divisia consumption index. Moreover, one may construct a dual index which is interpreted as price index. This index is called Konüs-Divisia price index. An interesting property of Konüs-Divisia price and consumption indices is that they are related through Yang transform (Shananin, 1986).

Revealed preference theory allows to test whether a finite set of observations on prices and consumption for a certain group of goods (this is called trade statistics) is consistent with Pareto demand theory. The criterion for the trade statistics to be consistent with Pareto theory with nonnegative, nonsatiated, continuous, concave, and monotonic utility function is generalized axiom of revealed preference (GARP). The criterion for the trade statistics to be consistent with Pareto theory with additional requirement for the utility function to be positively-homogeneous of degree 1 is homothetic axiom of revealed preference (HARP). If the trade statistics is consistent with Pareto demand theory then we say that it (or the corresponding group of goods) is rationalizable by a utility function. For treatment of the model of collective consumption when one explicitly model total household consumption as the result of decisions made by its members see (Cherchye

[2]See (Konüs, 1939) for English translation

et al., 2013, 2011, 2012). For developments in revealed preference techniques for applications other than rationalizability of consumer behavior on consumption commodities markets see (Carvajal et al., 2013; Echenique et al., 2013; Varian, 2012).

Rationalizability of the group means that it is full in terms of substitutes and complements. For any good in such group all the goods which are substitutes or complements to it should also belong to the group. The concept of rationalizabilty is related to the concept of separability. A subgroup of a rationalizable group is said to be separable if it is rationalizable. The separability allows one to build a structure of consumer demand.

The rationalizability of a group of goods should have some intuitive explanation. In other words one should be able to explain why a rationalizable group is full in terms of substitutes and complements. If almost any group of goods is rationalizable then the concept of rationalizability does not provide a tool for studying the segmentation of market, because it allows almost any structure of consumer demand. We show in this paper that this is the case for rationalizability based on GARP. More precisely, we show empirically that a randomly chosen group of goods satisfies GARP with high probability. Several researchers (see (Beatty and Crawford, 2011), for example) notice that one of the problem of low power of GARP is that the prices and consumption data are growing over time due to economic growth and this implies low number of budget sets intersections. This is the reason why in practice we often are unable to reject GARP.

The situation is different with HARP because this axiom is invariant with respect to changing of scales in economic data and the low number of intersections between budget sets does not lower the restrictiveness of HARP. We provide empirical evidence in favor of this conclusion by showing that the probability of having a random group (of not very large size) satisfying HARP is very low. This implies that if a group satisfies HARP then most probably it is not a mere coincidence and the group indeed may be interpreted as an independent market segment.

These observations led us to conjecture that one should use HARP as the instrument of revealing the true market segmentation. This means that one should consider rationalizability only by positively-homogeneous utility functions. In further sections we provide more empirical evidence in favor of importance of the requirement for the utility function to be positively-homogeneous by studying the power of tests based both on GARP and HARP.

If a trade statistics does not satisfy GARP and/or HARP there may be two different reasons for this. The first reason is that the selected group of goods is not complete and the consumer behavior cannot be explained by means of the concept of rational representative consumer. The second possible cause is the measurement error in data. In order to differ these two reasons one needs to have a quantitative measure of the degree to which the trade statistics satisfies revealed preference axioms. The first such measure was suggested by (Afriat, 1973). After that there were many papers devoted to alternative ways of measuring the degree of violation of trade statistics with GARP (see (Echenique et al., 2011; Ekeland and Galichon, 2013; Famulari, 1995; Gross, 1995; Smeulders et al., 2013; Varian, 1990; Whitney and Swofford, 1987)).

The power of GARP was assessed in (Bronars, 1987). The author generated random

consumption data and then checked how often the data set constructed from original prices and randomized consumption satisfied GARP. The drawback of the author's approach is that he ignores correlation in time series of consumption. He generated random consumption data independently for each period. We eliminate this drawback by using random process in generating randomized data sets. We have not seen empirical papers on the power of HARP. We also have not seen any study of the probability of having a random group satisfying HARP and GARP.

The two tests in revealed preference theory allow one to predict consumption at some fixed price. The approach which is due to (Varian, 1982) is to find all possible consumption bundles which when combined with the given data set constitute rationalizable trade statistics. The similar approach is used to predict prices given some consumption bundle. In applied research one prefers to have such set of possible price vectors to be as small as possible. We provide empirical evidence showing that the approach based on GARP may result in very large sets which are close to the set of all vectors with positive elements. However, the approach based on HARP provide much smaller sets for the same groups of commodities.

We measure the size of the set of predicted prices as the probability of having a trade statistics with the last period price being a random vector to satisfy revealed preference axiom (GARP or HARP). This approach is close to the measure of restrictiveness of GARP suggested by (Beatty and Crawford, 2011) who also suggest the measure of success in testing GARP which takes into account the strictness of the constraints implied by GARP. They also report low level of restrictiveness of GARP constraints from their empirical study. These findings are consistent with our findings about the size of the set of predicted prices implied by GARP.

Having a set of Engel curves we say that this set is rationalizable in some class of utility functions if any trade statistics consisting of the prices defining the Engel curves and set of demands on these curves (one demand from each curve) is rationalizable in the same class of utility functions. The problem of testing a set of arbitrary Engel curves for consistency with GARP is still an open one. However, in case when Engel curves are rays this problem has solution. Namely, in order to test a set such Engel curves for consistency with GARP one needs to test a single trade statistics built from these curves for consistency with HARP. For arbitrary set of Engel curves, even without intersections and monotone, it is not enough to check a single trade statistics for consistency with GARP to prove their consistency with GARP.

One may use the concept of rationalizability of Engel curves in order to build a smaller forecasting set for prices or consumption. In case of HARP, however, this does not lead to narrowing of the forecasting set. This may make forecasting set based on consistency with GARP smaller. However, since there is no way of checking for consistency of Engel curves with GARP[3], there is no way to construct this forecasting set even numerically. An attempt to provide an algorithm for building this set was made by (Blundell et al., 2008, 2014). However, the method suggested there is incorrect. We formulate the main

[3]Except for checking all possible trade statistics built from the set Engel curves for consistency with GARP.

result of (Blundell et al., 2008) in subsection 2.4 and prove its incorrectness by providing a counter-example in appendix 2.

The rest of the paper is organized in the following way. In section 2 we describe the two tests for consistency of data with utility maximization hypothesis, their generalizations on the case of data with measurement errors, how to forecast prices or consumption, and the notions of separability and hierarchy of index numbers[4]. Section 3 contains examples of applications of Konüs-Divisia index numbers to real data. In section 4 we estimate the power of the tests, and in section 5 we assess the sizes of forecasting sets based on the considered tests from revealed preference theory. Section 6 concludes. Appendix 1 contains proofs for yet unpublished results. Appendix 2 provides counter-example showing incorrectness of theorem 1 in (Blundell et al., 2008).

2 Theory

In this section we describe the instruments for testing data for consistency with maximization behavior and forecasting. We also describe a quantitative measures of the degree to what a data set fails to satisfy the hypothesis of maximization behavior. The methods of forecasting and building a tree of economic indices are also discussed.

2.1 Testing data for consistency with utility maximization hypothesis

Consider a group of m goods. Denote a consumption bundle of these goods by $X = (X_1, \dots, X_m)$ and their prices by $P = (P_1, \dots, P_m)$. A finite set $\{(P^t, X^t)\}_{t=1}^T$ of observations on prices and consumption in periods $1, \dots, T$ is called trade statistics. We say that it is rationalizable in functional class Φ if there exists a utility function $u \in \Phi$ such that

$$X^t \in \operatorname{Argmax} \left\{ u(X) \,\middle|\, \left\langle P^t,\, X \right\rangle \leqslant \left\langle P^t,\, X^t \right\rangle, X \in \mathbb{R}^m_+ \right\} \quad \forall t \in \{1, \dots, T\},$$

where $\langle \cdot,\, \cdot \rangle$ denotes scalar product and $\mathbb{R}^m_+ = \{X \in \mathbb{R}^m \mid X \geqslant 0, X \neq 0\}$.

In this paper we consider two classes of utility functions. The first one is that of nonnegative, nonsatiated, continuous, concave, and monotonic on $\mathbb{R}^m_+$ and positive on $\operatorname{int} \mathbb{R}^m_+$ functions. We denote it by Φ_G. The second one is that of functions from Φ_G which are positively-homogeneous of degree 1. The second class is denoted as Φ_H.

The trade statistics $\{(P^t, X^t)\}_{t=1}^T$ may be tested for rationalizability in these two classes by means of the two Afriat-Varian theorems.

Theorem 1. *(Afriat, 1963, 1967; Varian, 1983) The following statements are equivalent*

1) trade statistics $\{(P^t, X^t)\}_{t=1}^T$ is rationalizable in Φ_G;

[4]This also is known as utility tree.

2) there exist numbers U^t, $\lambda^t > 0$, $(t = 1, \ldots, T)$, such that

$$U^t \leqslant U^s + \lambda^s \left(\left\langle P^s, X^t \right\rangle - \left\langle P^s, X^s \right\rangle \right), \quad \forall t, s = 1, \ldots, T; \tag{1}$$

3) trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies GARP, which means that for all t and s if there exists $t_1, \ldots, t_k$ such that $\left\langle P^t, X^t \right\rangle \geqslant \left\langle P^t, X^{t_1} \right\rangle$, $\left\langle P^{t_1}, X^{t_1} \right\rangle \geqslant \left\langle P^{t_1}, X^{t_2} \right\rangle$, $\ldots$, $\left\langle P^{t_k}, X^{t_k} \right\rangle \geqslant \left\langle P^{t_k}, X^s \right\rangle$, then $\left\langle P^s, X^s \right\rangle \leqslant \left\langle P^s, X^t \right\rangle$;

4) the function $F_G(X) = \min_{s \in \{1,\ldots,T\}} \{U^s + \lambda^s (\langle P^s, X \rangle - \langle P^s, X^s \rangle)\}$, where $\{U^s, \lambda^s\}_{s=1}^T$ satisfy (1) and $\lambda^s > 0$ $\forall s \in \{1, \ldots, T\}$, rationalizes trade statistics.

In order to check whether the trade statistics satisfies GARP one needs to build a transitive closure of the relation R defined on $\{X^t\}_{t=1}^T \times \{X^t\}_{t=1}^T$ as

$$X^t R X^s \Leftrightarrow \left\langle P^t, X^t \right\rangle \geqslant \left\langle P^t, X^s \right\rangle. \tag{2}$$

If the resulting transitive closure R^* is such that for all $t, s \in \{1, \ldots, T\}$ $X^t R^* X^s$ implies $\left\langle P^s, X^s \right\rangle \leqslant \left\langle P^s, X^t \right\rangle$, then the trade statistics satisfies GARP. See (Varian, 1982) for an algorithm which allows one to construct a solution to the system of linear inequalities (1) if the trade statistics satisfies GARP.

Theorem 2. *(Afriat, 1963, 1967; Varian, 1983) The following statements are equivalent*

1) trade statistics $\{(P^t, X^t)\}_{t=1}^T$ is rationalizable in Φ_H;

2) there exist numbers $\lambda^t > 0$ $(t = 1, \ldots, T)$, such that

$$\lambda^t \left\langle P^t, X^s \right\rangle \geqslant \lambda^s \left\langle P^s, X^s \right\rangle, \quad \forall t, s = 1, \ldots, T; \tag{3}$$

3) trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies Homothetic Axiom of Revealed Preference (HARP), which means that for all subsets of indices $\{t_1, \ldots, t_k\}$ from $\{1, \ldots, T\}$ the following inequality is satisfied:

$$\left\langle P^{t_1}, X^{t_2} \right\rangle \left\langle P^{t_2}, X^{t_3} \right\rangle \ldots \left\langle P^{t_k}, X^{t_1} \right\rangle \geqslant \left\langle P^{t_1}, X^{t_1} \right\rangle \left\langle P^{t_2}, X^{t_2} \right\rangle \ldots \left\langle P^{t_k}, X^{t_k} \right\rangle. \tag{4}$$

4) the function $F(X) = \min_{s \in \{1,\ldots,T\}} \{\lambda^s \langle P^s, X \rangle\}$, where $\{\lambda^s\}_{s=1}^T$ satisfy (3) and $\lambda^s > 0$ $\forall s \in \{1, \ldots, T\}$, rationalizes trade statistics.

The inequality (4) for $k = 2$ may be written as

$$\frac{\left\langle P^{t_1}, X^{t_2} \right\rangle}{\left\langle P^{t_2}, X^{t_2} \right\rangle} \geqslant \frac{\left\langle P^{t_1}, X^{t_1} \right\rangle}{\left\langle P^{t_2}, X^{t_1} \right\rangle}.$$

This inequality reflects Gerschenkron's effect – Laspeyres index is no less than Paasche index (t_2 is base period, t_1 is current period).

A solution to the system of linear inequalities (3) may be found by means of Floyd-Warshall algorithm (Floyd, 1962; Warshall, 1962). Introduce Paasche price indices matrix C with elements given by

$$C_{ts} = \frac{\langle P^s,\, X^s \rangle}{\langle P^t,\, X^s \rangle}.$$

Then the system (3) may be rewritten as

$$\lambda^s C_{ts} \leqslant \lambda^t. \tag{5}$$

Let C^*_{ts} be equal to the maximum value of $C_{tt_1} C_{t_1 t_2} \dots C_{t_k s}$ over all possible ordered subsets $\{t_1, \dots, t_k\}$ of $\{1, \dots, T\}$ for any positive integer k assuming that the empty subset corresponds to C_{ts}, that is

$$C^*_{ts} = \max\{C_{tt_1} C_{t_1 t_2} \dots C_{t_k s} \mid \{t_1, \dots, t_k\} \subset \{1, \dots, T\}, k \geqslant 0\}.$$

Trade statistics satisfies HARP if and only if $C^*_{tt} \leqslant 1\ \forall t \in \{1, \dots, T\}$. One may notice that if the system (5) has positive solution, then it is equivalent to

$$\lambda^s C^*_{ts} \leqslant \lambda^t. \tag{6}$$

Consider an idempotent semi-ring with operations $a \oplus b = \max(a, b)$ and $a \otimes b = ab$. Then the matrix C^* with the elements C^*_{ts} is given by

$$C^* = C \oplus C^{\bullet 2} \oplus \dots \oplus C^{\bullet k} \oplus \dots, \tag{7}$$

where $C^{\bullet n}$ means taking n-th power of the matrix C in idempotent sense with all summing operations replaced by $\oplus$. Notice that all the elements of the matrix C are positive. If at some step $n \leqslant T$ of taking idempotent powers we have $C^*_{tt} > 1$ for some t, then all the elements of C^* are equal to ∞ and the system (6) has no solution. Otherwise, it is enough to take the first T terms in (7) in order to compute C^*. Therefore, the complexity of algorithm for computation of C^* is of order T^3. If the system (6) has a positive solution then

$$\lambda^t = \max\{C^*_{t\tau} \mid \tau \in \{1, \dots, T\}\} \quad t = 1, \dots, T$$

solve (5), which means that these λ^t solve (3).

Given a positive solution of the system (3) one may define Konüs consumption index as $F(X) = \min\{\lambda^s \langle P^s,\, X \rangle \mid s \in \{1, \dots, T\}\}$. This index is also Divisia index. That is why we call it as Konüs-Divisia consumption index. The corresponding Konüs-Divisia price index is given by function $Q(P)$ which is Yang transform of $F(X)$, that is

$$Q(P) = \inf\left\{ \frac{\langle P,\, X \rangle}{F(X)} \,\middle|\, X \geqslant 0, F(X) > 0 \right\}.$$

This transform comes from the optimization problem

$$\max\left\{F(Y) \,\middle|\, \langle P(X),\, Y \rangle \leqslant \langle P(X),\, X \rangle,\, Y \in \mathbb{R}^m_+ \right\},$$

where Lagrange multiplier to the constraint $\langle P(X),\, Y\rangle \leqslant \langle P(X),\, X\rangle$ is given by $\frac{1}{Q(P(X))}$. Yang transform is involutive in Φ_H (see (Kleiner, 1980)). In (Vratenkov and Shananin, 1991) it was suggested to compute consumption and price Konüs-Divisia index numbers as $F(X^t) = \lambda^t \left\langle P^t,\, X^t\right\rangle$ and $Q(P^t) = \frac{1}{\lambda^t}$.

This approach to construction of index numbers is called nonparametric one (see (Diewert, 1973; Houtman, 1995; Shananin, 1993)).

The relation between GARP and HARP is given by

Theorem 3. *(Shananin, 2009) The following statements are equivalent:*

1) *trade statistics* $\{(P^t, X^t)\}_{t=1}^T$ *satisfies HARP;*

2) *trade statistics* $\{(P^t, \mu^t X^t)\}_{t=1}^T$ *satisfies GARP for any positive values of* μ^t $(t \in \{1, \ldots, T\})$.

In order to discuss one of the implications of this theorem we need to define the concept of rationalizability of the set of Engel curves. Consider a set of Engel curves $\{q^t(\cdot)\}_{t=1}^T$ corresponding to price vectors $\{P^t\}_{t=1}^T$. This set of Engel curves is said to be rationalizable in Φ if there exists a function $u \in \Phi$ rationalizing trade statistics $\{(P^t, q^t(x_t))\}_{t=1}^T$ for any positive total expenditures x_1, …, x_T. Theorem 3 implies that if a trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies HARP then the set of Engel curves $\{q^t(\cdot)\}_{t=1}^T$ corresponding to prices $\{P^t\}_{t=1}^T$ is rationalizable in Φ_G. The fact that it is also rationalizable in Φ_H comes from the invariance of HARP with respect to change of scales in consumption data. Indeed, if the trade statistics satisfies HARP then it is rationalizable by positively-homogeneous function from Φ_G. This implies that all Engel curves are rays. Therefore, choosing trade statistics corresponding to a different values of expenditures from the set of Engel curves is equivalent to the change of scales in consumption data.

The invariance of HARP with respect to change of scales in prices and consumption data is a very important property. It makes rationalizability in Φ_H a more reliable property since trade statistics data are subject to changes of scales in time caused by economic growth and inflation.

2.2 Testing inverse demand functions for consistency with utility maximization hypothesis and the hierarchy of Konüs-Divisia indices

In this subsection we provide modified versions of theorems 1 and 2 for the case when we know the inverse demand functions $P(X)$ rather than their values at finite set of consumption bundles. We assume that the inverse demand functions are continuous on $\mathbb{R}_+^m$.

The relation R given by (2) is called a direct revealed preference relation. It was introduced in (Samuelson, 1938) in case when we have inverse demand function $P(X)$ as

$$XRY \Leftrightarrow \langle P(X),\, X\rangle \geqslant \langle P(X),\, Y\rangle\,.$$

We say that the inverse demand functions are rationalizable in functional class Φ if there exists a utility function $u \in \Phi$ such that

$$X \in \operatorname{Argmax}\left\{u(Y) \,\middle|\, \langle P(X),\, Y\rangle \leqslant \langle P(X),\, X\rangle\,, Y \in \mathbb{R}_+^m\right\}, \forall X \in R_+^m.$$

Theorem 4. *(Stigum, 1973) Let $P(X)$ be nonnegative, continuous on $\mathbb{R}_+^m$ functions such that $\langle P(X),\, X\rangle > 0$ for all $X \in \mathbb{R}_+^m$. Then the following statements are equivalent*

1) *the inverse demand functions are rationalizable in Φ_G;*

2) *there exist continuous on $\operatorname{int}\mathbb{R}_+^m$ functions $U(X) \in \Phi_G$ and $\lambda(X)$ with $\lambda(X) > 0$ $\forall X \in \mathbb{R}_+^m$, such that*

$$U(X) \leqslant U(Y) + \lambda(Y)\left(\langle P(Y),\, X\rangle - \langle P(Y),\, Y\rangle\right) \quad \forall X \in \mathbb{R}_+^m, Y \in \mathbb{R}_+^m; \tag{8}$$

3) *the inverse demand functions satisfy GARP, that is if for some X, X^1, ..., X^K, Y from $\mathbb{R}_+^m$ $\langle P(X),\, X\rangle \geqslant \left\langle P(X),\, X^1\right\rangle$, $\left\langle P(X^1),\, X^1\right\rangle \geqslant \left\langle P(X^1),\, X^2\right\rangle$, ..., $\left\langle P(X^K),\, X^K\right\rangle \geqslant \left\langle P(X^K),\, Y\right\rangle$ then $\langle P(Y),\, Y\rangle \leqslant \langle P(Y),\, X\rangle$.*

Consider a function F from Φ_H. Let $\hat{P} \in \partial F(\hat{X})$, where $\partial F(\hat{X})$ is the superdifferential of the function F at point $\hat{X}$. Then as shown in (Shananin, 2009) the generalized Euler identity

$$F(\hat{X}) = \left\langle \hat{P},\, \hat{X}\right\rangle$$

holds true.

Let us look at the inequalities (8). They imply that Y solves the optimization problem

$$\begin{aligned} &U(X) \to \max_X, \\ &\text{s.t. } \langle P(Y),\, X\rangle \leqslant \langle P(Y),\, Y\rangle\,, \\ &\qquad X \in \mathbb{R}_+^m. \end{aligned}$$

The first order condition for this problem is given by

$$\lambda(Y)P(Y) \in \partial U(Y).$$

If we require $U \in \Phi_H$ then the generalized Euler identity implies

$$U(Y) = \lambda(Y)\,\langle P(Y),\, Y\rangle \quad \forall Y \in \mathbb{R}_+^m.$$

If we substitute this in (8) we obtain the following inequalities

$$\lambda(Y)\,\langle P(Y),\, X\rangle \geqslant \lambda(X)\,\langle P(X),\, X\rangle\,.$$

These observations provide intuition on the criteria for rationalizability in Φ_H.

Theorem 5. *(Levin, 1997; Pospelova and Shananin, 1998b) Let $P(X)$ be nonnegative, continuous on $\mathbb{R}^m_+$ functions such that $\langle P(X), X\rangle > 0$ for all $X \in \mathbb{R}^m_+$. Then the following statements are equivalent*

1) the inverse demand functions $P(X)$ are rationalizable in Φ_H;

2) the system of linear inequalities

$$\lambda(Y)\langle P(Y), X\rangle \geqslant \lambda(X)\langle P(X), X\rangle \quad (X \in \mathbb{R}^m_+, Y \in \mathbb{R}^m_+) \tag{9}$$

has a solution $\lambda(X)$ which is positive and continuous on $int\,\mathbb{R}^m_+$;

3) the inverse demand functions $P(X)$ satisfy HARP, that is for any set of vectors $\{X^1, \dots, X^T\}$ from $\mathbb{R}^m_+$ the inequality

$$\left\langle P(X^1), X^2\right\rangle \left\langle P(X^2), X^3\right\rangle \dots \left\langle P(X^T), X^1\right\rangle \geqslant \left\langle P(X^1), X^1\right\rangle \left\langle P(X^2), X^2\right\rangle \dots \left\langle P(X^T), X^T\right\rangle$$

holds true;

4) there exist price index function $Q(P)$ and consumption index function $F(X)$ both from Φ_H such that

$$Q(P)F(X) \leqslant \langle P, X\rangle \qquad \forall P \in \mathbb{R}^m_+, X \in \mathbb{R}^m_+, \tag{10}$$
$$Q(P(X))F(X) = \langle P(X), X\rangle \qquad \forall X \in \mathbb{R}^m_+. \tag{11}$$

Corollary 1. *If $Q(P)$ and $F(X)$ are functions from Φ_H satisfying (10) and (11) then $\lambda(X) = \frac{1}{Q(P(X))}$ satisfies (9).*

Corollary 2. *The functions $Q(P(X))$ and $F(X)$ in 4) may be expressed through a solution $\lambda(X)$ to the system (9) as*

$$F(X) = \lambda(X)\langle P(X), X\rangle,$$
$$Q(P(X)) = \frac{1}{\lambda(X)}.$$

See appendix 1 for the proofs of these corollaries.

Now we move to the concept of separability. Assume that we have a group of goods split into $K+1$ subgroups and there are m_k goods in k-th subgroup (still m is the total number of goods so that $m = \sum_{k=1}^{K+1} m_k$). Let the elements of the consumption vector X be rearranged to reflect this split. Namely, assume that $X = (X_1', \dots, X_K', X_{K+1}')'$, where X_k is the consumption vector for the goods from k-th subgroup. Let the inverse demand functions $P(X)$ be rearranged in the same manner:

$$P(X) = \left((P_1(X))', \dots, (P_K(X))', (P_{K+1}(X))'\right)'.$$

The subgroups $1, \ldots, K$ are said to be separable from the original group if the group is rationalizable and the rationalizing function is represented as

$$F(X) = F_0(F_1(X_1), \ldots, F_K(X_K), X_{K+1}),$$

where all the functions F_0, F_1, …, F_K are from the same functional class Φ.

Theorem 6. *(Vratenkov and Shananin, 1991) Let the inverse demand function $P(X)$ be rationalizable in Φ_H and the subgroups $1, \ldots, K$ be separable from the original group so that the rationalizing function is represented as*

$$F(X) = F_0(F_1(X_1), \ldots, F_K(X_K), X_{K+1}),$$

where F_0, F_1, …, $F_K \in \Phi_H$. Then the Yang transform of $F(X)$

$$Q(P) = \inf\left\{ \frac{\langle P,\, X\rangle}{F(X)} \,\middle|\, X \geqslant 0, F(X) > 0 \right\}$$

may be represented as

$$Q(P) = Q_0(Q_1(P_1), \ldots, Q_K(P_K), P_{K+1}),$$

where P_k is price vector for the goods from the k-th subgroup, $Q_0(q_1, q_2, \ldots, q_K,\ P_{K+1})$ is the optimal value of objective function in the following optimization problem

$$\begin{aligned} &\frac{\sum\limits_{k=1}^{T} q_k Y_k + \langle P_{K+1},\, X_{K+1}\rangle}{F_0(Y_1, Y_2, \ldots, Y_K, X_{K+1})} \to \min, \\ &\text{s.t. } (Y_1, Y_2, \ldots, Y_K, X_{K+1}) \in R_+^{K+m_{K+1}}, \\ &\qquad F_0(Y_1, Y_2, \ldots, Y_K, X_{K+1}) > 0, \end{aligned}$$

and

$$Q_k(P_k) = \inf\left\{ \frac{\langle P_k,\, X_k\rangle}{F_k(X_k)} \,\middle|\, X_k \in \mathbb{R}_+^{m_k}, F_k(X_k) > 0 \right\} (k = 1, 2, \ldots, K).$$

Corollary 3. *Let $P_1(X), \ldots, P_K(X)$ be rationalizable in Φ_H. Let $Q_k(P_k)$ and $F_k(X_k)$ $(k = 1, \ldots, K)$ be Konüs-Divisia price and consumption indices that is for all $k = 1 \ldots, K$*

$$Q_k(P_k)F_k(X_k) \leqslant \langle P_k,\, X_k\rangle \qquad \forall P_k \in \mathbb{R}_+^{m_k}, X_k \in \mathbb{R}_+^{m_k}, \tag{12}$$

$$Q_k(P_k(X))F_k(X_k) = \langle P_k(X),\, X_k\rangle \qquad \forall X_k \in \mathbb{R}_+^{m_k}. \tag{13}$$

Let the inverse demand functions

$$\mathcal{P}(X) = (Q_1(P_1(X)), \ldots, Q_K(P_K(X)), P_{K+1}(X))$$

be rationalizable in Φ_H that is there exist functions $F_0(y_1, \ldots, y_K, X_{K+1})$ and $Q_0(q_1, \ldots,$ $q_K,\, P_{K+1})$ from Φ_H such that

$$Q_0(q_1, \ldots, q_K, P_{K+1})F_0(y_1, \ldots, y_K, X_{K+1}) \leqslant \sum_{k=1}^{K} q_k y_k + \langle P_{K+1},\, X_{K+1}\rangle\,, \tag{14}$$

for all $(q_1, \ldots, q_K) \in \mathbb{R}^K_+$, $(y_1, \ldots, y_K) \in \mathbb{R}^K_+$, $P_{K+1} \in \mathbb{R}^{m_{K+1}}_+$, $X_{K+1} \in \mathbb{R}^{m_{K+1}}_+$, and

$$Q_0(Q_1(P_1(X)), \ldots, Q_K(P_K(X)), P_{K+1}(X)) \\ \times F_0(F_1(X_1), \ldots, F(X_K), X_{K+1}) = \sum_{k=1}^{K+1} \langle P_k(X),\, X_k\rangle\,, \tag{15}$$

for all $X \in \mathbb{R}^m_+$. Then the inverse demand functions $P(X)$ are rationalizable in Φ_H. Moreover, the Konüs-Divisia price and consumption indices for the whole group of goods are represented as

$$F(X) = F_0(F_1(X_1), \ldots, F_K(X_K), X_{K+1}), \tag{16}$$

$$Q(P(X)) = Q_0(Q_1(P_1(X)), \ldots, Q_K(P_K(X)), P_{K+1}(X)). \tag{17}$$

See appendix 1 for the proof.

These results may easily be reformulated for trade statistics rather than inverse demand functions. Namely, let $TS = \{(P^t, X^t)\}_{t=1}^T$ be trade statistics for the whole group of goods with $X^t = ((X_1^t)', \ldots (X_K^t)', (X_{K+1}^t)')'$ and $P^t = ((P_1^t)', \ldots, (P_K^t)', (P_{K+1}^t)')'$. Assume that trade statistics for the first K subgroups satisfy HARP. Denote the corresponding Konüs-Divisia price and consumption indices as $Q_k(P_k^t)$ and $F_k(X_k^t)$. Construct new trade statistics $\mathcal{TS} = \{(\mathcal{P}^t, \mathcal{X}^t)\}_{t=1}^T$ where

$$\mathcal{P}^t = (Q_1(P_1^t), \ldots, Q_K(P_K^t), P_{K+1}^t), \quad \mathcal{X}^t = (F_1(X_1^t), \ldots, F_K(X_K^t), X_{K+1}^t).$$

Theorem 6'. *(Vratenkov and Shananin, 1991) If the trade statistics TS is rationalizable in Φ_H and corresponding Konüs-Divisia consumption indices are represented as*

$$F(X^t) = F_0(F_1(X_1^t), \ldots, F_K(X_K^t), X_{K+1}^t).$$

Then the corresponding Konüs-Divisia price indices may be represented as

$$Q(P^t) = Q_0(Q_1(P_1^t), \ldots, Q_K(P_K^t), P_{K+1}^t),$$

where $Q_0(q_1^t, q_2^t, \ldots, q_K^t,\, P_{K+1}^t)$ is the optimal value of objective function in the following optimization problem

$$\frac{\sum\limits_{k=1}^{T} q_k^t Y_k + \left\langle P_{K+1}^t,\, X_{K+1}\right\rangle}{F_0(Y_1, Y_2, \ldots, Y_K, X_{K+1})} \to \min,$$
$$\text{s.t. } (Y_1, Y_2, \ldots, Y_K, X_{K+1}) \in R_+^{K+m_{K+1}},$$
$$F_0(Y_1, Y_2, \ldots, Y_K, X_{K+1}) > 0,$$

and

$$Q_k(P_k^t) = \inf \left\{ \frac{\langle P_k^t,\, X_k \rangle}{F_k(X_k)} \,\middle|\, X_k \in \mathbb{R}_+^{m_k}, F_k(X_k) > 0 \right\} (k = 1, 2, \ldots, K).$$

Corollary 3'. *(Vratenkov and Shananin, 1991) If the trade statistics $\mathcal{TS}$ satisfies HARP then the original trade statistics TS also satisfies HARP. Moreover, Konüs-Divisia index numbers $F(X^t)$ and $Q(P^t)$ may be represented as*

$$F(X^t) = F_0(F_1(X_1^t), \ldots, F_K(X_K^t), X_{K+1}^t),$$
$$Q(P^t) = Q_0(Q_1(P_1^t), \ldots, Q_K(P_K^t), P_{K+1}^t).$$

The corollary 3' shows that the Konüs-Divisia indices of the original group are constructed from the Konüs-Divisia indices of the subgroups. The separable subgroups may be further split into smaller subgroups. When Konüs-Divisia index numbers are constructed in this way we say that we have a hierarchy of Konüs-Divisia index numbers. It may be represented as a graph in which the vertices correspond to groups of goods and edges correspond to the relation of separability. Such graph is one way of visualizing the structure of consumer demand. An example of such visualization is shown in subsection 3.1. In similar way one may build a hierarchy of economic indices based on GARP[5].

2.3 Irrationality indices

The trade statistics may fail to satisfy HARP or GARP because of errors in measurement of consumption and price data. This observation suggests that one needs some quantitative measure of the degree to what the trade statistics fails HARP or GARP.

Here we describe one approach to such quantitative measure. First, let us introduce parametric families of axioms of revealed preference. We say that the trade statistics satisfies GARP(ω) if for all t and s the existence of $t_1, \ldots, t_k$ such that $\langle P^t,\, X^t \rangle \geqslant \omega \langle P^t,\, X^{t_1} \rangle$, $\langle P^{t_1},\, X^{t_1} \rangle \geqslant \omega \langle P^{t_1},\, X^{t_2} \rangle$, …, $\langle P^{t_k},\, X^{t_k} \rangle \geqslant \omega \langle P^{t_k},\, X^{s} \rangle$ implies the inequality $\langle P^s,\, X^s \rangle \leqslant \omega \langle P^s,\, X^t \rangle$. We say that the trade statistics satisfies HARP(ω) if for all subsets of indices $\{t_1, \ldots, t_k\}$ from $\{1, \ldots, T\}$ with $t_i \neq t_{i+1}$ for all $i \in \{1, \ldots, k-1\}$ the following inequality is satisfied:

$$\langle P^{t_1},\, X^{t_2} \rangle \langle P^{t_2},\, X^{t_3} \rangle \ldots \langle P^{t_k},\, X^{t_1} \rangle \geqslant \frac{1}{\omega^k} \langle P^{t_1},\, X^{t_1} \rangle \langle P^{t_2},\, X^{t_2} \rangle \ldots \langle P^{t_k},\, X^{t_k} \rangle .$$

The additional requirement that $t_i \neq t_{i+1}$ is added in order to make it possible for the trade statistics to satisfy HARP(ω) with $\omega < 1$. We did not put this requirement in the previous subsection because it would not have affected whether certain trade statistics satisfied HARP or not. The Afriat-Varian theorems from the previous subsection may be generalized in the following way.

[5]In this case we mean only consumption indices as there is no analog of theorem 6 for rationalizability in Φ_G showing the duality between consumption and price indices.

Theorem 7. *The following statements are equivalent*

1) *trade statistics* $\{(P^t, X^t)\}_{t=1}^T$ *satisfies GARP(ω).*

2) *There exist numbers* U^t, $\lambda^t > 0$, $(t = 1, \dots, T)$, *such that*

$$U^t \leqslant U^s + \lambda^s \left(\omega \left\langle P^s,\, X^t \right\rangle - \left\langle P^s,\, X^s \right\rangle\right), \quad \forall t, s = 1, \dots, T, \quad t \neq s. \tag{18}$$

The proof is given in appendix 1.

Theorem 8 (Houtman (1995))**.** *The following statements are equivalent*

1) *trade statistics* $\{(P^t, X^t)\}_{t=1}^T$ *satisfies HARP(ω);*

2) *there exist numbers* $\lambda^t > 0$ $(t = 1, \dots, T)$, *such that*

$$\omega\lambda^t \left\langle P^t,\, X^s \right\rangle \geqslant \lambda^s \left\langle P^s,\, X^s \right\rangle, \quad \forall t, s = 1, \dots, T, \quad t \neq s; \tag{19}$$

One may construct Konüs-Divisia index numbers using a solution to the system (19) by formulas from the previous subsection.

Now we may introduce the quantitative measures of irrationalizability of trade statistics. For rationalizability in Φ_G the measure is given by GARP irrationality index

$$\omega_G = \min\{\omega \mid \text{trade statistics satisfies GARP}(\omega)\}.$$

This approach is close to that of (Afriat, 1973) where a critical efficiency level[6] was used to measure the degree of inconsistency of trade statistics with GARP. For rationalizability in Φ_H the measure is given by HARP irrationality index

$$\omega_H = \min\{\omega \mid \text{trade statistics satisfies HARP}(\omega)\}.$$

The method of constructing Konüs-Divisia index numbers using a solution of the system (19) with $\omega = \omega_H$ is called generalized nonparametric one (see (Pospelova and Shananin, 1998b)).

The theorem 3 also may be generalized to the case of $\omega \neq 1$.

Theorem 9. *The following statements are equivalent:*

1) *trade statistics* $\{(P^t, X^t)\}_{t=1}^T$ *satisfies HARP(ω);*

2) *trade statistics* $\{(P^t, \mu^t X^t)\}_{t=1}^T$ *satisfies GARP(ω) for any positive values of* μ^t $(t \in \{1, \dots, T\})$.

See appendix 1 for the proof.

[6]It is equal to $\frac{1}{\omega_G}$.

2.4 Forecasting

Assume that we have some trade statistics $\{(P^t, X^t)\}_{t=1}^T$. There are two forecasting problems which may be solved with revealed preference theory. The first one is to predict P^{T+1} given X^{T+1}. The second one is to predict X^{T+1} given P^{T+1}. Since both problems have similar solution, we describe the details only for the second problem.

Assume that the trade statistics satisfies GARP(ω). Then we may construct the set of forecasted demand vectors as

$$K_G^\omega(P^{T+1}) = \left\{ X^{T+1} \middle| \{(P^t, X^t)\}_{t=1}^{T+1} \text{ satisfies GARP}(\omega) \right\}.$$

This approach is a generalization of that for GARP(1) suggested in (Varian, 1982).

In the similar way we may construct the set of forecasted demand vectors assuming that the trade statistics satisfies HARP(ω) as

$$K_H^\omega(P^{T+1}) = \left\{ X^{T+1} \middle| \{(P^t, X^t)\}_{t=1}^{T+1} \text{ satisfies HARP}(\omega) \right\}.$$

Theorem 10. *(Grebennikov and Shananin, 2009)*
Assume that the trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies HARP(ω) with $\omega \geqslant 1$. Let

$$C_{ts}^*(\omega) = \max\left\{ \omega^{-k-1} C_{tt_1} C_{t_1 t_2} \dots C_{t_{k-1} t_k} C_{t_k s} \mid \{t_1, \dots, t_k\} \subset \{1, \dots, T\},\, k \geqslant 0 \right\},$$

and

$$\gamma_s\left(P^{T+1}, \omega\right) = \min_{t \in \{1, \dots, T\}} \left\{ \frac{\omega^2}{C_{ts}^*(\omega)} \frac{\left\langle P^{T+1},\, X^t \right\rangle}{P^t X^t} \right\}.$$

Then

$$K_H^\omega(P^{T+1}) = \left\{ X \in \mathbb{R}_+^m \mid \gamma_s\left(P^{T+1}, \omega\right) \langle P^s,\, X \rangle \geqslant \left\langle P^{T+1},\, X \right\rangle \ \forall s \in \{1, \dots, T\} \right\}.$$

The values $C_{ts}^*(\omega)$ may be effectively computed in $O(T^3)$ operations by means of Floyd-Warshall algorithm.

We may fix not only new price vector at the value P^{T+1} but also the expenditure at some level x_{T+1}. Then, the corresponding forecasting sets are given by

$$K_G^\omega(P^{T+1}, x_{T+1}) = K_G^\omega(P^{T+1}) \cap \left\{ X \in \mathbb{R}_+^m \mid \left\langle P^{T+1},\, X \right\rangle = x_{T+1} \right\},$$
$$K_H^\omega(P^{T+1}, x_{T+1}) = K_H^\omega(P^{T+1}) \cap \left\{ X \in \mathbb{R}_+^m \mid \left\langle P^{T+1},\, X \right\rangle = x_{T+1} \right\}.$$

The analytical expression for $K_G^1(P^{T+1}, 1)$ may be found in (Varian, 1982).

One may propose forecasting sets based on the concept of rationalizability of Engel curves. Let the set of Engel curves $\{q^t(\cdot)\}_{t=1}^T$ corresponding to price vectors $\{P^t\}_{t=1}^T$ satisfy GARP(ω). The latter means that for any set of individual expenditures $\{x_t\}_{t=1}^T$

the trade statistics $\big\{(P^t, q^t(x_t))\big\}_{t=1}^T$ satisfies GARP(ω). Then one may suggest to build the forecasting set for demand at a fixed price vector P^{T+1} as

$$\widetilde{K}_G^\omega(P^{T+1}) = \left\{ X \in \mathbb{R}_+^m \;\middle|\; \begin{matrix} \{(P^t, q^t(x_t))\}_{t=1}^T \cup \{(P^{T+1}, X)\} \text{ satisfies GARP}(\omega) \\ \text{for any } x_1 > 0, \ldots, x_T > 0 \end{matrix} \right\}.$$

Denote the set $\widetilde{K}_G^\omega(P^{T+1}) \cap \{X \in \mathbb{R}_+^m \mid \langle P^{T+1}, X\rangle = x_{T+1}\}$ as $\widetilde{K}_G^\omega(P^{T+1}, x_{T+1})$.

Let us introduce also the set $\widetilde{K}_H^\omega(P^{T+1})$ as

$$\widetilde{K}_H^\omega(P^{T+1}) = \left\{ X \in \mathbb{R}_+^m \;\middle|\; \begin{matrix} \{(P^t, q^t(x_t))\}_{t=1}^T \cup \{(P^{T+1}, X)\} \text{ satisfies HARP}(\omega) \\ \text{for any } x_1 > 0, \ldots, x_T > 0 \end{matrix} \right\}.$$

It is worth noting that $\widetilde{K}_H^1(P^{T+1}) = K_H^1(P^{T+1})$. That is because if a trade statistics $\{(P^t, q^t(x_t))\}_{t=1}^T$ satisfies HARP for some positive total expenditures $x_1, \ldots, x_K$ then it is rationalizable with positively-homogeneous utility function. This implies that the Engle curves $q^t(\cdot)$ are rays. Therefore, choosing total expenditures different from $x_1, \ldots, x_K$ is equivalent to change of scales in consumption data. Since HARP is invariant to such changes, the trade statistics $\{(P^t, q^t(x'_t))\}_{t=1}^T$ satisfies HARP for any positive total expenditures $x'_1, \ldots, x'_K$.

A very close approach was studied by (Blundell et al., 2008). The difference is that the authors used Strong Axiom of Revealed Preference (SARP). It is similar to GARP though is little bit more restrictive. The trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies SARP if $X^t R^* X^s$ and $X^t \neq X^s$ implies $\langle P^s, X^s\rangle < \langle P^s, X^t\rangle$. The authors of (Blundell et al., 2008) consider the following support sets

$$S(P^{T+1}, x_{T+1}, \{x_t\}_{t=1}^T) = \\ \left\{ X^{T+1} \in \mathbb{R}_+^m \;\middle|\; \begin{matrix} \langle P^{T+1}, X^{T+1}\rangle = x_{T+1}, \\ \{(P^t, q^t(x_t))\}_{t=1}^T \cup \{(P^{T+1}, X)\} \text{ satisfies SARP,} \end{matrix} \right\}.$$

Similarly, one may consider support sets based on GARP:

$$G(P^{T+1}, x_{T+1}, \{x_t\}_{t=1}^T) = \\ \left\{ X^{T+1} \in \mathbb{R}_+^m \;\middle|\; \begin{matrix} \langle P^{T+1}, X^{T+1}\rangle = x_{T+1}, \\ \{(P^t, q^t(x_t))\}_{t=1}^T \cup \{(P^{T+1}, X)\} \text{ satisfies GARP,} \end{matrix} \right\}.$$

The authors define intersection demands $\tilde{X}^t = q^t(\tilde{x}_t)$ as

$$\langle P^{T+1}, q^t(\tilde{x}_t)\rangle = x_{T+1},$$

and claim that if demands are weakly normal[7] then for any $\{x_t\}_{t=1}^T$

$$S(P^{T+1}, x_{T+1}, \{\tilde{x}_t\}_{t=1}^T) \subseteq S(P^{T+1}, x_{T+1}, \{x_t\}_{t=1}^T). \tag{20}$$

[7]This means that if $x > x'$ then $q^t(x) \geqslant q^t(x')$ for all $t = \overline{1, T}$.

Since the closure of $S(P^{T+1}, x_{T+1}, \{x_t\}_{t=1}^T)$ is equal to $G(P^{T+1}, x_{T+1}, \{x_t\}_{t=1}^T)$, this implies that $\widetilde{K}_G^1(P^{T+1}, x_{T+1}) = G(P^{T+1}, x_{T+1}, \{\tilde{x}_t\}_{t=1}^T)$. This conclusion is incorrect. We provide counterexample showing its incorrectness in Appendix 2.

The further decrease in the size of the set K_H^ω is possible if we add the additional requirement for the trade statistics to satisfy the Law of Demand. This law states that the price index is decreasing function of the demand index. We say that the trade statistics satisfies the Law of Demand if it may be continued to the demand functions satisfying the Law of Demand.

Let us define $\widehat{K}_H^\omega$ by

$$\widehat{K}_H^\omega = \left\{ X^{T+1} \middle| \{(P^t, X^t)\}_{t=1}^{T+1} \text{ satisfies HARP}(\omega) \text{ and the Law of Demand} \right\}.$$

The analytical expression for $\widehat{K}_H^\omega$ is not known yet. However, (Grebennikov and Shananin, 2009) provides external estimate for this set.

Theorem 11. *(Grebennikov and Shananin, 2009)*

Assume that the trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies HARP(ω) with $\omega \geqslant 1$. Let

$$D_{st} = \max \left\{ \frac{\langle P^t, X^t \rangle}{\omega \langle P^s, X^t \rangle}, \Theta\left(\frac{\langle P^s, X^s \rangle}{\langle P^s, X^t \rangle} - \omega \right) \right\}, \tag{21}$$

where

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leqslant 0, \end{cases}$$

and

$$\Delta_{st} = \max_{\{t_1,\dots,t_k\} \subset \{1,\dots,T\}} \left\{ D_{st_1} D_{t_k t} \prod_{i=2}^{k} D_{t_{i-1} t_i} \right\}.$$

Then

$$\widehat{K}_H^\omega \subset \left\{ X^{T+1} \in \mathbb{R}_+^m \middle| \max \left[\frac{\langle P^{T+1}, X^{T+1} \rangle}{\omega \langle P^s, X^{T+1} \rangle}, \Theta\left(\frac{\langle P^s, X^s \rangle}{\langle P^s, X^{T+1} \rangle} - 1 \right) \right] \right. \\ \left. \times \max \left[\frac{\langle P^t, X^t \rangle}{\omega \langle P^{T+1}, X^t \rangle}, \Theta\left(\frac{\langle P^{T+1}, X^{T+1} \rangle}{\omega \langle P^{T+1}, X^t \rangle} - 1 \right) \right] \leqslant \frac{1}{\Delta_{st}}, \quad \forall s, t = \overline{1, T} \right\}.$$

3 Applications of Konüs-Divisia index numbers

In this section we describe several empirical applications of Konüs-Divisia index numbers introduced in 2.1. We used two data sets. The first one contains consumption and price data for 196 goods in Hungary for years 1975-1984 (Kozponti Statistikal Hivantal, 1985). During these years Hungarian economy was shifting from plan to market one and

it is interesting to analyze the changes in the structure of consumer demand happened in this period. The second one contains similar data for 106 goods in the Netherlands for years 1951-1977 (Netherlands. Centraal Bureau voor de Statistiek. Hoofdafdeling Statistische Methoden, 1981). These are post-war years. The both data sets covers the period when middle class was emerging.

3.1 Trade statistics of Hungary

In figure 1 we show prices and consumption for six meat products from Hungarian trade statistics (in thin lines) as well as Konüs-Divisia consumption and price indices (in thick lines). We see that Konüs-Divisia index numbers are less volatile compared to raw prices and consumption. This implies that they may be forecasted more accurately than prices and consumption of goods themselves.

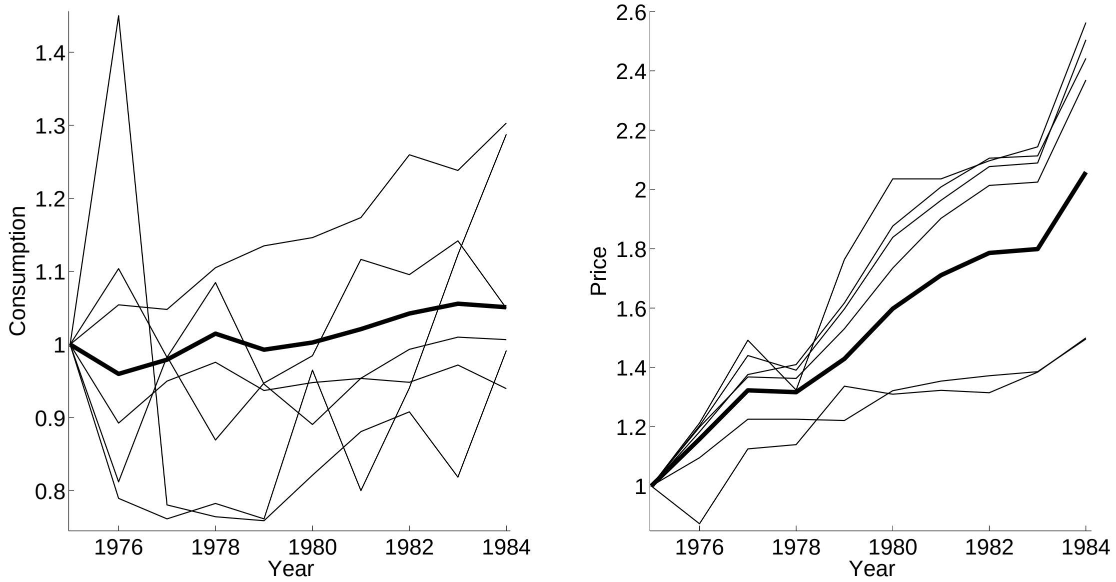

Figure 1: Prices, consumption, and Konüs-Divisia indices for group of meat product.

The commodities in Hungarian trade statistics are split by Hungary trade experts into 11 groups shown in table 1. The commodity groups differ by the duration of commodity usage. The first three groups represent the everyday goods which are consumed in 1 to 3 months. The goods from "Clothes" groups are used during a year on average. All the rest groups contain goods and services of durable consumption with characteristic duration of service of 5-10 years.

The time series of Konüs-Divisia indices for the groups of all goods, of everyday goods, and of goods and services of durable consumption are shown in figure 2. We see that the

Table 1: Commodity groups in the trade statistics of Hungary.

Gr. number	Group name	Num. of goods
1	Consumption goods	49
2	Beverages	15
3	Tobacco goods	3
4	Clothes	31
5	House service	5
6	Heating, home energy	12
7	Home equipment	30
8	Health care, hygiene	7
9	Transport, information	11
10	Education, culture, sport, recreation	23
11	Other goods	10

demand was shifting from everyday goods to goods and services of durable consumption. The growth rate of the consumption of goods and services of durable consumption was higher that of everyday goods. This shift may be attributed to formation of middle class in Hungary during the 1974-1984. This change in social structure lead to the change of the hierarchy of economic indices.

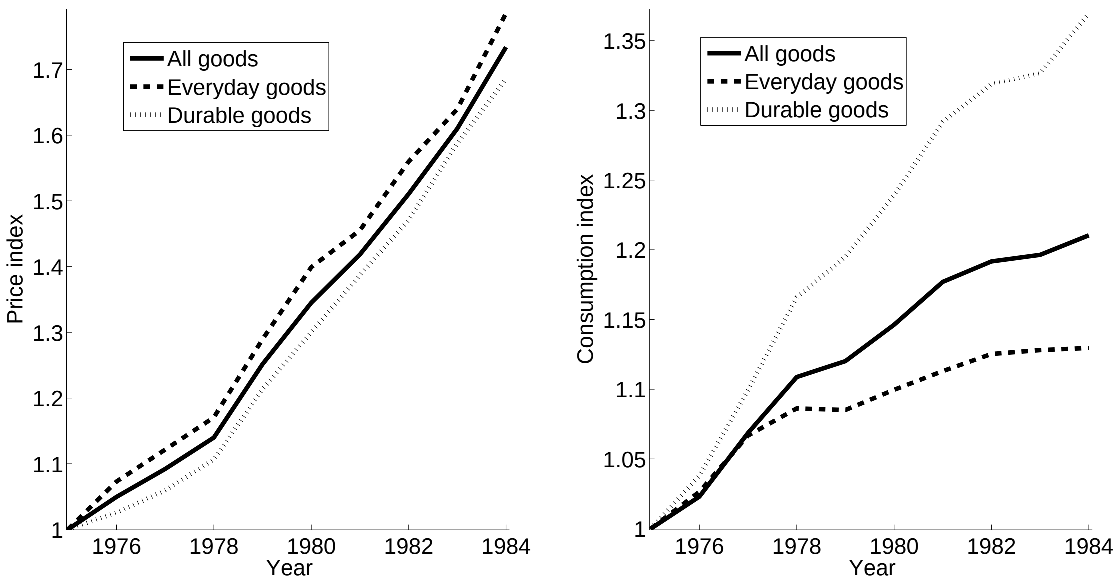

Figure 2: Segmentation in Hungarian trade statistics.

Out of the 11 groups selected by Hungary trade experts only the first two satisfy

HARP. The group formed by all goods also satisfies HARP. The fact that the majority of the commodity groups selected by Hungary trade experts does not satisfy HARP implies that such classification does not reflect the structure of consumer demand in Hungary during 1975-1984.

A different classification based on characteristic time of consumption was suggested in (Vratenkov and Shananin, 1991). The goods from the first three groups from table 1 form a group which satisfies HARP. These goods form new aggregated good which is called as "everyday goods" in (Vratenkov and Shananin, 1991). The goods from the groups from 5 to 11 in table 1 also form a group satisfying HARP. The corresponding aggregated good is named "goods and services of durable consumption" in (Vratenkov and Shananin, 1991). The group "Clothes" does not satisfy HARP, but if did then we would have another aggregated good called "goods with medium time of consumption". However, the group with all the goods from the group "Clothes" and the aggregated good "everyday goods" does satisfy HARP. The group formed by the group "Clothes" plus the aggregated good "goods and services of durable consumption" also satisfies HARP. Moreover, both these groups prove to be separable from the group formed by all the goods. The latter group also satisfies HARP. This structure of consumer demand was revealed in (Vratenkov and Shananin, 1991) and is shown in figure 3.

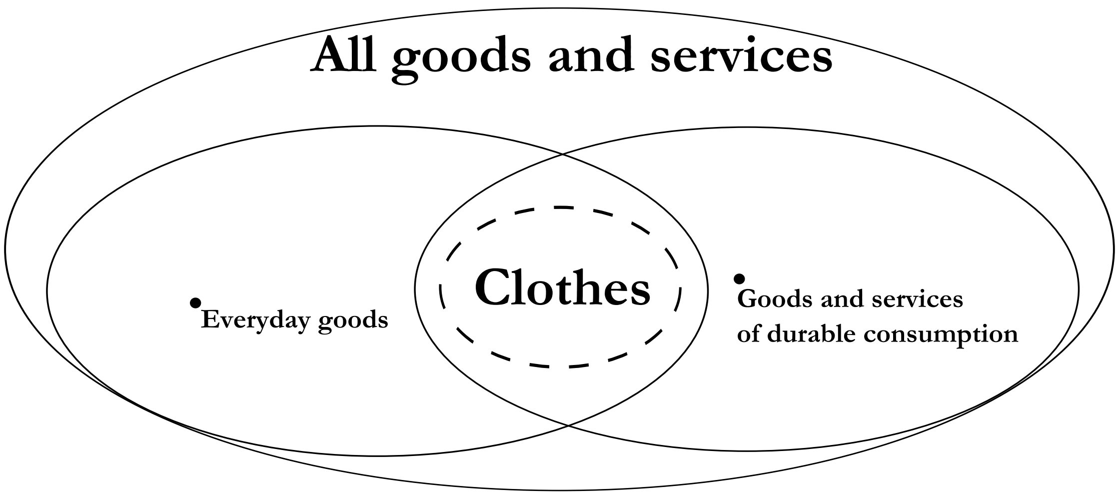

Figure 3: The structure of consumer demand in Hungary during 1975-1984.

More details on studying this statistic may be found in (Kondrakov et al., 2010; Pospelova and Shananin, 1998b; Vratenkov and Shananin, 1991).

We used this statistics in order to asses the probability of having a randomly chosen group satisfying GARP or HARP for various sizes of random groups. More precisely, for each size of random groups in range from 3 to 193 we generated 100000 random groups[8] of goods and computed the fraction of them satisfying GARP and HARP. The values of

[8]The number of different groups of sizes 2, 194, 195, and 196 is less than 100000 so we

computed fractions are plotted against the sizes of random groups in figure 4. We see that for any size of random group the probability that it satisfies GARP is close to 1. This means that there will be a lot of groups which are complete in terms of substitutes and compliments if we use GARP in order to build the hierarchy of economic indices. This means that GARP allows almost any structure of consumer demand and does not lead to revealing the true one.

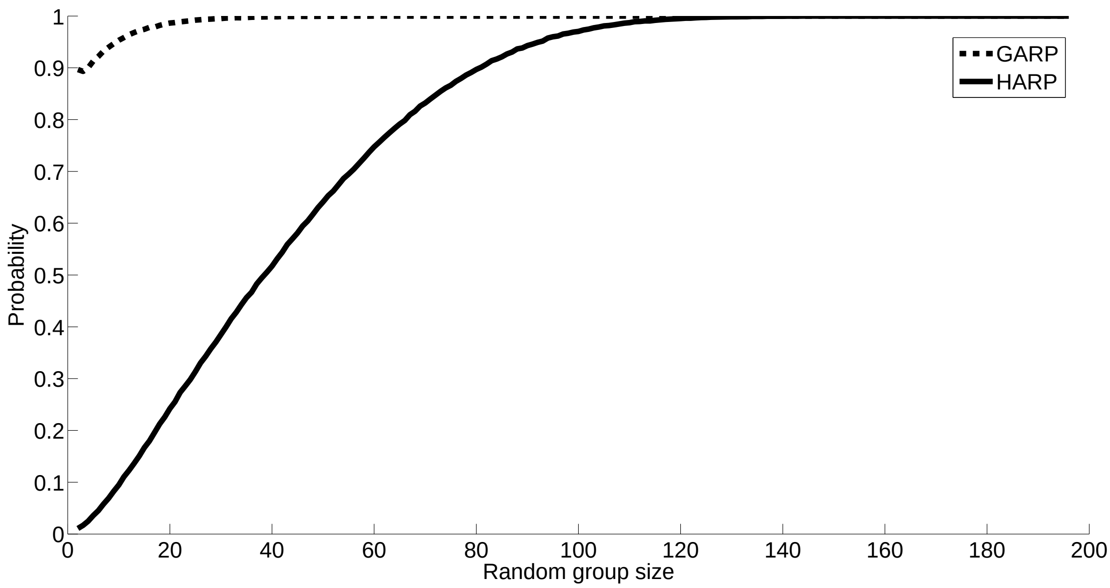

Figure 4: Probability of having random groups satisfying HARP and GARP on statistics of the Hungary.

The situation is different for HARP. Although almost all large groups of goods satisfy HARP[9], the majority of small groups does not satisfy HARP. This means that a group satisfying HARP is more valuable and meaningful for the construction of the tree of economic indices since there is more evidence in favor of it being an independent market segment.

3.2 Trade statistics of the Netherlands

This trade statistics was analyzed in (Pospelova and Shananin, 1998a). Dutch trade experts split all the goods into five groups. None of them satisfy HARP. However the

considered all of them. We did not considered groups consisting of a single good because any such group satisfies both GARP and HARP.

[9]This is because the group containing all the goods satisfies HARP and exclusion of small number of goods leads only to slight violations of HARP in terms of HARP irrationality index.

group of all goods does satisfy HARP. The commodity groups with corresponding irrationality indices are shown in table 2.

Table 2: Commodity groups in the trade statistics of the Netherlands.

Gr. number	Goods	Num. of goods	ω_H
0	All goods	106	0.9998
1	Grocery, milk, meat, fish	42	1.0002
2	Sweets, tobacco goods, beer	12	1.0034
3	Clothes, cars, home equipment	21	1.0073
4	Heating, water, medicine	11	1.0002
5	Renting, service	20	1.0167

Two ways of evaluating Konüs-Divisia indices for the group of all goods were considered in (Pospelova and Shananin, 1998a). The first way is to compute indices for the groups selected by trade experts using generalized nonparametric method, then form new trade statistics of the resulting Konüs-Divisia indices and evaluate Konüs-Divisia indices for it. Denote price indices computed in this way by $Q_1(P^t)$. The second way is to apply nonparametric method directly to the group of all goods without intermediate aggregation. Denote price indices computed in this way as $Q_2(P^t)$.

The time series of these two price indices were obtained in (Pospelova and Shananin, 1998a) and are shown in figure 5. We see that there is little difference between them. Indeed, the maximum deviation between $Q_1(P^t)$ and $Q_2(P^t)$ is 1.76%. This provides evidence in favor of robustness of Konüs-Divisia indices to methods of aggregation.

We repeated the empirical study of the probability of having random group satisfying HARP or GARP from the previous subsection on the statistics of the Netherlands. The estimated probabilities are shown in figure 6. Since in this statistics we have more than twice time periods than in the statistics of Hungary we expected the probabilities become lower. For HARP this is indeed so. We see that the probability of having random group satisfying HARP does not reach the value of one for any random group size below the total number of goods. The probability of having a random group satisfying GARP became lower for small random groups. However, it is still extremely close to one for random groups of sizes greater than 30.

4 Power of HARP and GARP tests

In this section we assess the power of tests for consistency with maximization behavior using the trade statistics of Hungary. The general approach is the following. Consider a trade statistics $\{(P^t, X^t)\}_{t=1}^T$. Denote random trade statistics with random prices and original consumption as $\widetilde{TS} = \{(\tilde{P}^t, X^t)\}_{t=1}^T$. We generate B realizations of $\widetilde{TS}$: $\{\widetilde{TS}_b\}_{b=1}^B$. For each generated trade statistics we evaluate irrationality indices $\omega_{G,b} =$

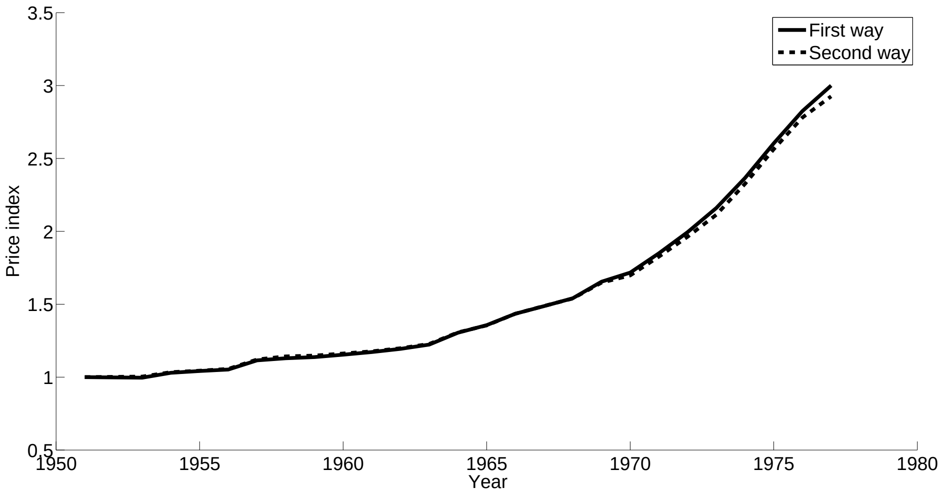

Figure 5: Konüs-Divisia price indices constructed in two different ways.

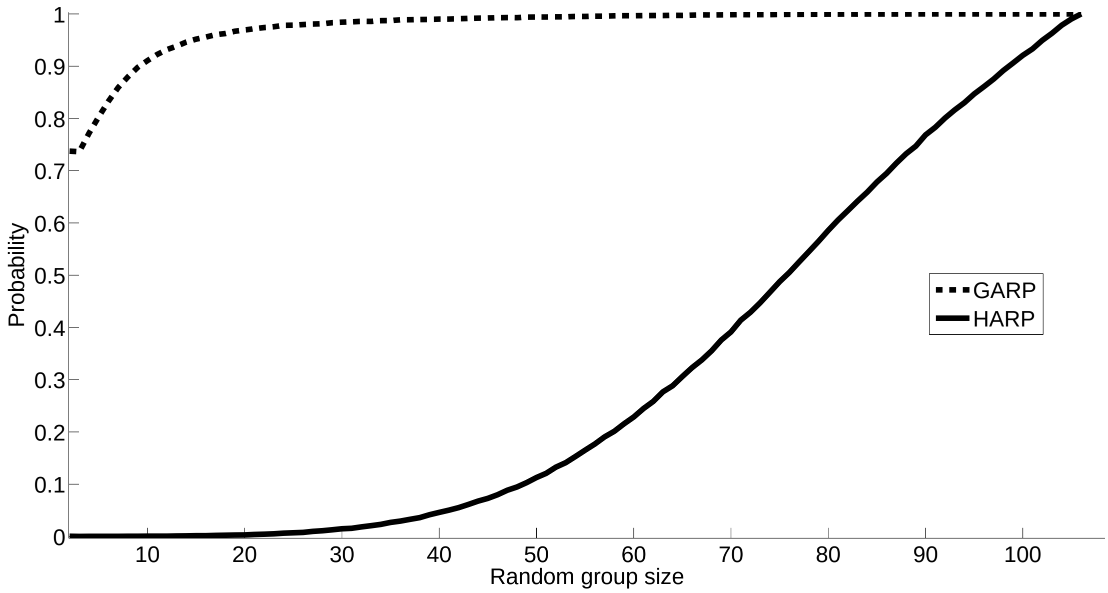

Figure 6: Probability of having random groups satisfying HARP and GARP on statistics of the Netherlands.

$\omega_G(\widetilde{TS}_b)$ and $\omega_{H,b} = \omega_H(\widetilde{TS}_b)$. The power of GARP test is then estimated as

$$\widehat{W}_G = \frac{1}{B}\sum_{b=1}^{B} \mathbb{I}(\omega_{G,b} > 1),$$

where $\mathbb{I}(\omega_{G,b} > 1)$ is equal to 1 if $\omega_{G,b} > 1$ and equal to 0 otherwise. The power of HARP test is estimated by

$$\widehat{W}_H = \frac{1}{B}\sum_{b=1}^{B} \mathbb{I}(\omega_{H,b} > 1).$$

These estimates depend on the original trade statistics $\{(P^t, X^t)\}_{t=1}^T$ as well as the distribution of $(\tilde{P}^1, \dots, \tilde{P}^T)$. In this paper we model $\tilde{P}^t$ in the following way. First, we fit auto-regression model $AR(r_i)$ for $z_i^t = \log\left(\frac{P_i^t}{P_i^{t-1}}\right)$ for each good. We chose r_i from $\{0, 1, 2\}$ using Akaike information criterion (see, for example, (Hayashi, 2000)). We put the limit on possible values of r_i because the small number of time periods. The estimated model for the price of good i is

$$\begin{aligned} z_i^t &= \hat{\beta}_{i,0} + \sum_{\tau=1}^{\hat{r}_i} \hat{\beta}_{i,\tau} z_i^{t-\tau} + \varepsilon_{i,t}, \quad t \in \{\hat{r}_i + 2, \dots, T\}, \\ z_i^t &= \log\left(\frac{P_i^t}{P_i^{t-1}}\right), \quad t \in \{1, \dots, \hat{r}_i + 1\}, \end{aligned} \tag{22}$$

where $\varepsilon_{i,t} \sim N(0, \hat{\sigma}_i^2)$ are independent random variables. We put hats on the parameters of AR model as well as its order to emphasize that these are estimates. The distributions of $\tilde{P}^t$ are such that $\tilde{z}_i^t = \log\left(\frac{\tilde{P}_i^t}{\tilde{P}_i^{t-1}}\right)$ follow (22).

We took $B = 20000$ and estimated $\widehat{W}_G$ and $\widehat{W}_H$ for the groups selected by Hungarian trade experts[10] which satisfy HARP. These groups are shown in table 3 in the next section. The values of $\widehat{W}_H$ lie in $[0.97540, 0.99985]$. The values of $\widehat{W}_G$ lie in $[0, 0.53080]$. The histogram for $\widehat{W}_G$ is shown in figure 7. We see that the power of GARP test is very low compared to that of HARP test. The high power of HARP test implies that the fact that a certain group of goods satisfies HARP is a valuable finding, because it implies a strong relation between consumption and price. The low power of GARP test implies that the GARP often holds true even for group of goods with no relation between price and consumption. That is why GARP should not be used as a tool for studying the structure of consumer demand.

5 Size of forecasting set

In this section we use the trade statistics of Hungary in order to assess the size of the forecasting set for the last period price given the actual value of last period consumption. The approach used in this section is similar to that used in the previous section. The groups are also the same.

[10]The groups in table 1 represent the classification of goods; trade experts considered not only them.

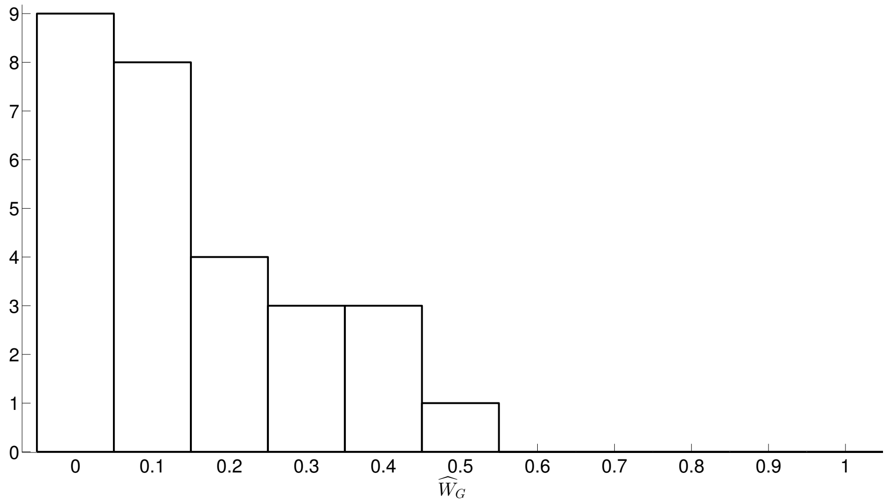

Figure 7: The histogram of $\widehat{W}_G$.

Consider random trade statistics $\widetilde{TS} = \{(\tilde{P}^t, X^t)\}_{t=1}^T$, where $\tilde{P}^t = P^t$ for $t \in \{1, \ldots, T-1\}$. The last period price is a random vector with uniformly distributed on $\{P \in \mathbb{R}^m_+ \mid \|P\| = 1\}$. The size of forecasting set is measured in the following way. We generate B realizations of $\widehat{TS}$: $\{\widetilde{TS}_b\}_{b=1}^B$. The size of forecasting set based on GARP is measured by

$$\widehat{F}_G = \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}(\widetilde{TS}_b \text{ satisfies GARP}),$$

where $\mathbb{I}(\widetilde{TS}_b \text{ satisfies GARP})$ is equal to 1 if $\widetilde{TS}_b$ satisfies GARP and is equal to 0 otherwise. The size of forecasting set based on HARP is measured similarly:

$$\widehat{F}_H = \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}(\widetilde{TS}_b \text{ satisfies HARP}).$$

These measures are consistent estimated for the probability of having a random price vector P in the forecasting set based on GARP and HARP respectively. The values of any of these measures close to 1 imply that the corresponding forecasting set is close to the set of all possible price vectors.

The estimated values of $\widehat{F}_G$ and $\widehat{F}_H$ for $B = 100000$ are shown in table 3. We see that for the majority of the considered groups the value of $\widehat{F}_G$ is equal to one. This shows that the forecasting set based on GARP often gives almost no information about future prices. The situation is different for the forecasting set based on HARP. We see that this forecasting set indeed provide some information on future prices as it puts certain restrictions on them.

Table 3: The sizes of forecasting sets for selected groups from Hungarian trade statistics.

Group description	$\widehat{F}_H$	$\widehat{F}_G$
Meat + meat products	0.29239	0.76349
Same + fish products	0.2763	0.71422
Foodstuff	0.27244	1
Same + sweets + fruits and vegetables + beverages + tobacco	0.15899	1
(1-67) + clothes + housing service + energy in home + furnishings and home equipment	0.1535	1
Same + health and hygiene	0.15713	1
Same + transport and information	0.15538	1
Same + education, culture, sport and recreation	0.15969	1
All goods	0.16237	1

6 Conclusion

In this paper we have demonstrated an approach to construction of economic indices based of homothetic axiom of revealed preference. We have provided examples of computing these indices and building the tree of economic indices which serves as a visualization of the structure of consumer demand. We demonstrated that testing data for consistency with this axiom is a better approach to studying the structure of consumer demand that based on generalized axiom of revealed preference. We provided two arguments for this.

The first argument is that there is high probability of having a random group satisfying GARP. This means that the analysis based on GARP reveals that almost any group is full in terms of substitution and complementarity. Such outcome is useless and contradicts intuition. Most probably by choosing absolutely random group we should reveal that it does not represent an independent market segment. The analysis based on HARP allows one to choose full groups of not very great number of goods more accurately which makes it proper tool for studying the market segmentation.

The second argument is that the power of GARP test is very low compared to that of HARP test. We compared the power of HARP and GARP test on the groups which satisfy HARP. The fact that these groups satisfy HARP provides additional evidence in favor of their completeness. However, the power of GARP test on these groups is very low. This means that GARP test does not catch strong relation between prices and consumption of the goods in the groups despite we have strong evidence in favor of existence of such relation.

We also have demonstrated that HARP test is suitable for forecasting future prices. The resulting forecasting set is not a trivial one. This is not the case for GARP test. The forecasting set based on GARP is close to a trivial one. Moreover, in many cases it coincides with it.

Appendix 1

Proof of corollary 1. The relations (10) and (11) imply that for all $X \in \operatorname{int} \mathbb{R}^m_+$ and $P \in \mathbb{R}^m_+$ such that $Q(P) > 0$

$$\frac{1}{Q(P(X))} \langle P(X),\, X \rangle \leqslant \frac{1}{Q(P)} \langle P,\, X \rangle \,.$$

The division by $Q(P(X))$ is correct because if $X \in \operatorname{int} \mathbb{R}^m_+$ then $F(X) > 0$n and $\langle P(X),\, X \rangle > 0$. Therefore, (11) implies $Q(P(X)) > 0$. Substituting P with $P(Y)$ leads to

$$\frac{1}{Q(P(X))} \langle P(X),\, X \rangle \leqslant \frac{1}{Q(P(Y))} \langle P(Y),\, X \rangle \quad \forall X, Y \in \operatorname{int} \mathbb{R}^m_+.$$

Therefore, $\lambda(X) = \frac{1}{Q(P(X))}$ satisfies (9). Since $Q(P(X))$ is continuous so is $\lambda(X)$.

Q.E.D.

Proof of corollary 2. Define $F(X)$ and $Q(P)$ by

$$F(X) = \inf \left\{ \lambda(Y) \langle P(Y),\, X \rangle \,\middle|\, Y \in \mathbb{R}^m_+ \right\}, Q(P) = \inf \left\{ \frac{\langle P,\, Y \rangle}{F(Y)} \,\middle|\, Y \geqslant 0, F(Y) > 0 \right\},$$

where $\lambda(Y)$ satisfies (9). Then the functions $F(X)$ and $Q(P)$ satisfy (10). Since $\lambda(Y)$ satisfies (9),

$$F(X) = \lambda(X) \langle P(X),\, X \rangle \,,$$

and

$$\begin{aligned} Q(P(X)) &= \inf \left\{ \frac{\langle P(X),\, Y \rangle}{\lambda(Y) \langle P(Y),\, Y \rangle} \,\middle|\, Y \geqslant 0, \lambda(Y) \langle P(Y),\, Y \rangle > 0 \right\} \\ &= \frac{1}{\lambda(X)} \inf \left\{ \frac{\lambda(X) \langle P(X),\, Y \rangle}{\lambda(Y) \langle P(Y),\, Y \rangle} \,\middle|\, Y \geqslant 0, \lambda(Y) \langle P(Y),\, Y \rangle > 0 \right\} = \frac{1}{\lambda(X)}. \end{aligned}$$

Q.E.D.

Proof of corollary 3. We show that functions $F(X)$ and $Q(P)$ defined as

$$\begin{aligned} F(X) &= F_0(F_1(X_1), \ldots, F_K(X_K), X_{K+1}), \\ Q(P) &= Q_0(Q_1(P_1), \ldots, Q_K(P_K), P_{K+1}) \end{aligned}$$

are from Φ_H and satisfy (10) and (11). The fact that these functions are from Φ_H is implied from the definition of functions Q_k and F_k ($k = 1, \dots, K+1$). The inequality (10) follows from (12) and (14):

$$Q(P)F(X) = Q_0(Q_1(P_1), \dots, Q_K(P_K), P_{K+1})F_0(F_1(X_1), \dots, F_K(X_K), X_{K+1})$$
$$\leqslant \sum_{k=1}^{K} Q_k(P_k)F_k(X_k) + \langle P_{K+1},\, X_{K+1}\rangle \leqslant \langle P,\, X\rangle$$

The identity (11) follows from (15). Theorem 5 implies that the inverse demand functions $P(X)$ are rationalizable in Φ_H. The identities (16) and (17) are satisfied by definition of functions $F(X)$ and $Q(P)$.

Q.E.D.

Proof of theorem 7. 1) $\Rightarrow$ 2). This part represent a modified version of the part of the proof of theorem 1 given in (Varian, 1982). Let $\max(I)$ be the index of maximal element with respect to the binary relation $R^*(\omega)$ which is the transitive closure of binary relation $R(\omega)$ defined on $\{X^t\}_{t=1}^T \times \{X^t\}_{t=1}^T$ as

$$X^t R(\omega) X^\tau \Leftrightarrow \left\langle P^t,\, X^t \right\rangle \geqslant \omega \left\langle P^t,\, X^\tau \right\rangle.$$

In other words,

$$\forall t \in I \quad X^t R^*(\omega) X^{\max(I)} \Rightarrow X^{\max(I)} R^*(\omega) X^t.$$

Consider the following algorithm.
Output: A set of numbers U^t, $\lambda^t > 0$ ($t = \overline{1,T}$).

1) $I = \{1, \dots, T\}$, $B = \varnothing$.
2) Let $m = \max(I)$.
3) Set $E = \{t \in I \mid X^t R^*(\omega) X^m\}$. If $B = \varnothing$, set $U^m = \lambda^m = 1$ and go to 6). Otherwise go to 4).
4) Set $U^m = \min\limits_{t\in E} \min\limits_{\tau \in B} \min \left\{U^\tau + \lambda^\tau \left(\omega \left\langle P^\tau,\, X^t\right\rangle - \left\langle P^\tau,\, X^\tau\right\rangle\right), U^\tau\right\}$.
5) Set $\lambda^m = \max\limits_{t\in E} \max\limits_{\tau \in B} \max \left\{\dfrac{U^\tau - U^m}{\omega \left\langle P^t,\, X^\tau\right\rangle - \left\langle P^t,\, X^t\right\rangle}, 1\right\}$.
6) Set $U^t = U^m$, $\lambda^t = \lambda^m$ for all $t \in E$.
7) Set $I = I \setminus E$, $B = B \cup E$. If $I = \varnothing$, stop. Otherwise, go to 2).

Let us prove that if trade statistics satisfies GARP(ω), then this algorithm provides the solution to (18). The algorithm is an iterative process. We show that after each iteration of step 6) the constructed U's and λ's satisfy corresponding inequalities of (18). Namely, we show that

$$\begin{aligned}
(a)\quad & U^t \leqslant U^\tau + \lambda^\tau \left(\omega \left\langle P^\tau, X^t \right\rangle - \left\langle P^\tau, X^\tau \right\rangle\right) \quad \forall \tau \in B,\, t \in E, \\
(b)\quad & U^\tau \leqslant U^t + \lambda^t \left(\omega \left\langle P^t, X^\tau \right\rangle - \left\langle P^t, X^t \right\rangle\right) \quad \forall \tau \in B,\, t \in E, \\
(c)\quad & U^t \leqslant U^\tau + \lambda^\tau \left(\omega \left\langle P^\tau, X^t \right\rangle - \left\langle P^\tau, X^\tau \right\rangle\right) \quad \forall t, \tau \in E,\, t \neq \tau.
\end{aligned}$$

Proof of (a): By step 4) of the algorithm:

$$U^t = U^m \leqslant U^\tau + \lambda^\tau \left(\omega \left\langle P^\tau, P^t \right\rangle - \left\langle P^\tau, X^\tau \right\rangle\right) \quad \forall \tau \in B,\, t \in E.$$

Proof of (b): For this we need to use step 5). Notice that $\omega \left\langle P^t, X^\tau \right\rangle > \left\langle P^t, X^t \right\rangle$ for all $\tau \in B$. If not, then $X^t R^*(\omega) X^\tau$ for some $\tau \in B$. But then t would have been moved into B before τ was and we have contradiction with $t \in E$.

Hence, the division is well defined and

$$\lambda^t = \lambda^m \geqslant \frac{U^\tau - U^t}{\omega \left\langle P^t, X^\tau \right\rangle - \left\langle P^t, X^t \right\rangle} \quad \forall \tau \in B,\, t \in E.$$

Cross multiplying:

$$\lambda^t \left(\omega \left\langle P^t, X^\tau \right\rangle - \left\langle P^t, X^t \right\rangle\right) \geqslant U^\tau - U^t \quad \forall \tau \in B,\, t \in E,$$

which proves (b).

Proof of (c): Note that if $t, \tau \in E$, then $\omega \left\langle P^\tau, X^t \right\rangle \geqslant \left\langle P^\tau, X^\tau \right\rangle$. If not, then $\left\langle P^\tau, X^\tau \right\rangle > \omega \left\langle P^\tau, X^t \right\rangle$, which contradicts GARP(ω). Indeed, $t \in E$ implies $X^t R^*(\omega) X^m$, $\tau \in E$ implies $X^\tau R^*(\omega) X^m$ which (by definition of m) implies $X^m R^*(\omega) X^\tau$, so $X^t R^*(\omega) X^\tau$ and GARP(ω) implies $\left\langle P^\tau, X^\tau \right\rangle \leqslant \omega \left\langle P^t, X^\tau \right\rangle$. Now for all $t, \tau \in E$:

$$U^t = U^\tau, \quad \lambda^\tau = \lambda^m > 0,$$

so

$$U^t \leqslant U^\tau + \lambda^\tau \left(\omega \left\langle P^\tau, X^t \right\rangle - \left\langle P^\tau, X^\tau \right\rangle\right).$$

2) $\Rightarrow$ 1). Let $\{U^t, \lambda^t\}_{t=1}^T$ with $\lambda^t > 0, \forall t \in \{1, \ldots, T\}$ satisfy (18), and $\left\langle P^t, X^t \right\rangle \geqslant \omega \left\langle P^t, X^{t_1} \right\rangle$, $\left\langle P^{t_1}, X^{t_1} \right\rangle \geqslant \omega \left\langle P^{t_1}, X^{t_2} \right\rangle$, ..., $\left\langle P^{t_k}, X^{t_k} \right\rangle \geqslant \omega \left\langle P^{t_k}, X^s \right\rangle$. Taking into account that $\lambda^t > 0\ \forall t \in \{1, \ldots, T\}$, these inequalities imply that $U^t \leqslant U^s$. This implies $\left\langle P^s, X^s \right\rangle \leqslant \omega \left\langle P^s, X^t \right\rangle$ which completes the proof of this part.

Q.E.D.

Proof of theorem 9. 1) $\Rightarrow$ 2). HARP(ω) is invariant with respect to change of scales. Therefore if the trade statistics $\{(P^t, X^t)\}_{t=1}^T$ satisfies HARP(ω) then the trade statistics $\{(P^t, \mu^t X^t)\}_{t=1}^T$ also satisfies HARP(ω) for any μ^t ($t \in \{1, \ldots, T\}$). This implies that $\{(P^t, \mu^t X^t)\}_{t=1}^T$ satisfies GARP(ω).

2) $\Rightarrow$ 1). Fix some subset of indices $(t_1, \ldots, t_k) \subset \{1, \ldots, T\}$ such that there are no two identical indices. Select μ^{t_1}, …, μ^{t_k} so that

$$\begin{aligned}
\left\langle P^{t_1},\, \mu^{t_1} X^{t_1} \right\rangle &= \omega \left\langle P^{t_1},\, \mu^{t_2} X^{t_2} \right\rangle, \\
\left\langle P^{t_2},\, \mu^{t_2} X^{t_2} \right\rangle &= \omega \left\langle P^{t_2},\, \mu^{t_3} X^{t_3} \right\rangle, \\
&\dots \\
\left\langle P^{t_{k-1}},\, \mu^{t_{k-1}} X^{t_{k-1}} \right\rangle &= \omega \left\langle P^{t_{k-1}},\, \mu^{t_k} X^{t_k} \right\rangle.
\end{aligned}$$

It is possible to do that because there are $k-1$ equalities to satisfy by choosing k numbers. Then GARP(ω) implies

$$\left\langle P^{t_k},\, \mu^{t_k} X^{t_k} \right\rangle \leqslant \omega \left\langle P^{t_k},\, \mu^{t_1} X^{t_1} \right\rangle.$$

Therefore,

$$\frac{\left\langle P^{t_1},\, X^{t_2} \right\rangle}{\left\langle P^{t_1},\, X^{t_1} \right\rangle} \frac{\left\langle P^{t_2},\, X^{t_3} \right\rangle}{\left\langle P^{t_2},\, X^{t_2} \right\rangle} \cdots \frac{\left\langle P^{t_k},\, X^{t_1} \right\rangle}{\left\langle P^{t_k},\, X^{t_k} \right\rangle} \geqslant \frac{1}{\omega^k}.$$

Q.E.D.

Appendix 2

Consider a group of three goods the demand on which is given by the three Engel curves

$$\begin{aligned}
q_1(x) &= (1,\, \varepsilon,\, \varepsilon)' x, \\
q_2(x) &= (\varepsilon,\, 1,\, \varepsilon)' x, \\
q_3(x) &= (\varepsilon,\, \varepsilon,\, 1)' x,
\end{aligned}$$

which correspond to the following price vectors:

$$\begin{aligned}
P^1 &= (2\,1\,4)', \\
P^2 &= (2\,1\,2)', \\
P^3 &= (2\,2\,1)'.
\end{aligned}$$

Observed demands are given by $X^t(\varepsilon) = q_t(1)$. The matrix $PX(\varepsilon)$ with elements $px_{\tau t}(\varepsilon) = \left\langle P^\tau,\, X^t(\varepsilon) \right\rangle$ is equal to

$$PX(\varepsilon) = \begin{pmatrix} 2+5\varepsilon & 1+6\varepsilon & 4+3\varepsilon \\ 2+3\varepsilon & 1+4\varepsilon & 2+3\varepsilon \\ 2+3\varepsilon & 2+3\varepsilon & 1+4\varepsilon \end{pmatrix}.$$

If $\varepsilon = 0$ then the direct revealed preference relation is given by

$$R(0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with transitive closure $R^*(0) = R(0)$. The trade statistics $\{(P^t, X^t(0))\}_{t=1}^T$ satisfies GARP and HARP. In order to simplify the calculations we set $\varepsilon = 0$. Having zero demand may seem unnatural, however the forecasting sets for small enough positive ε should differ only a little unless setting $\varepsilon > 0$ leads to failure of GARP[11]. It may be shown that $R(\varepsilon) = R(0)$ for $\varepsilon < 1$. Therefore, choosing ε from $(0, 1)$ does not lead to failure of GARP.

Let $P^4 = (1\,1\,1)'$ and $x_4 = 2$. Since Engel curves are rays, $K_H^1(P^4, x_4) = \widetilde{K}_G^1(P^4, x_4)$. Therefore, if $\widetilde{K}_G^1(P^4, x_4) = G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3)$, then $G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3) = K_H^1(P^4, x_4)$. We show that this is not true.

We start with $G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3)$. The intersection demands are given by

$$\left\langle P^4,\, q_t(\tilde{x}_t) \right\rangle = \tilde{x}_t = 2.$$

and the set $\{X \in \mathbb{R}_+^m \mid X \in G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3)$ is given by

$$\begin{aligned} &X_1 \leqslant 2 - \frac{3}{2} X_2 \\ &X_3 = 2 - X_1 - X_2 \\ &X_1 \geqslant 0, X_2 \geqslant 0. \end{aligned}$$

The set $\{X \in \mathbb{R}_+^m \mid X \in K_H^1(P^4, x_4)$ is given by

$$\begin{aligned} &X_3 = 2 - X_1, \\ &X_2 = 0, \\ &X_1 \geqslant 0,\ X_1 \leqslant 2. \end{aligned}$$

We see that the set $K_H^1(P^4, x_4)$ is a proper subset of $G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3)$. This implies that (20) is not true. If we set $\varepsilon = 0.5$ then $G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3)$ is been given by

$$\begin{aligned} &X_1 \leqslant 1.75 - 1.5 X_2 \\ &X_2 \leqslant 1 \\ &X_1 + X_2 \geqslant 1 \\ &X_3 = 2 - X_1 - X_2 \\ &X_1 \geqslant 0,\ X_2 \geqslant 0, \end{aligned}$$

[11] This is not a problem for HARP since this axiom is a fixed set of non-strict inequalities.

while the set $K_H^1(P^4, x_4)$ is given by

$$
\begin{aligned}
&X_1 \leqslant 1.75 - 1.5X_2 \\
&X_2 \leqslant 0.625 \\
&X_1 + X_2 \geqslant 1 \\
&X_3 = 2 - X_1 - X_2 \\
&X_1 \geqslant 0,\ X_2 \geqslant 0.
\end{aligned}
$$

The set $K_H^1(P^4, x_4)$ is still a proper subset of $G(P^4, x_4, \{\tilde{x}_t\}_{t=1}^3)$.

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